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An integral equation of Muskhelishvili type: Strong quantum electrodynamics

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An equation that arises out of the bifurcation analysis of an improvement of the nonperturbative equations for the electron mass function in quenched quantum electrodynamics is analyzed. In the quasilinear approximation, the integral equation is solved by Mellin transformation, followed by the calculation of the Muskhelishvili index of the resultant singular integral operator. © 1995 American Institute of Physics.

In recent years several improvements of the ladder approximation in gauge theories have been suggested. The most ambitious one is perhaps that of Bashir and Pennington,¹ in which a criticism by Dong, Munczek, and Roberts² of an earlier work³ had been constructively incorporated into a new fermion–gauge boson vertex *Ansatz*. A general conclusion that may be drawn is that the above *Ansätze* display a satisfactory insensitivity to the gauge parameter:⁴ indeed the Bashir–Pennington form is constructed to yield a strictly gauge-independent critical coupling for chiral symmetry breakdown.

The fact that gauge covariance seems to be well in hand suggests that one may as well use the Landau gauge for actual calculations, in which considerable simplification takes place. This had indeed earlier often been done, in the framework of the ladder approximation, although with less justification, since there the gauge dependence is very troublesome. In fact, in the Landau gauge, the linearized forms of both the Curtis–Pennington and the more recent Bashir–Pennington equations for the electron propagator in quantum electrodynamics are the same.

In order to examine the onset of chiral symmetry breaking, one considers the bifurcation equation,^{5,6} which amounts to a linearization of the Dyson–Schwinger equations with respect to the mass function. The result is

$$\mathcal{M}(x) = \frac{\alpha}{4\pi} \int_0^{\Lambda^2} \frac{dy}{y + \mathcal{M}^2(y)} J(y, x), \quad (1)$$

where α is the fine-structure coupling constant, and where to the first order in the mass function, \mathcal{M} , the kernel has the form

$$J(y, x) = 3 \left[\frac{y}{x} \theta(x - y) + \theta(y - x) \right] \mathcal{M}(y) - 3x \frac{\mathcal{M}(y) - \mathcal{M}(x)}{y - x} \left[\frac{y^2}{x^2} \theta(x - y) + \theta(y - x) \right]. \quad (2)$$

The true bifurcation equation is obtained by also linearizing the denominator in Eq. (1), i.e., by replacing $y + \mathcal{M}^2(y)$ there by y . This yields

$$\mathcal{M}(x) = \frac{3\alpha}{4\pi} \int_0^\infty dy \left\{ \frac{\theta(x-y)}{x} \left(\mathcal{M}(y) - \frac{y}{2} \frac{\mathcal{M}(y) - \mathcal{M}(x)}{y-x} \right) + \frac{\theta(y-x)}{y} \left(\mathcal{M}(y) - \frac{x}{2} \frac{\mathcal{M}(y) - \mathcal{M}(x)}{y-x} \right) \right\}. \quad (3)$$

Note that the ultraviolet cutoff, Λ , has been taken to infinity, which is appropriate at the bifurcation point.⁴ This equation is solved by

$$\mathcal{M}(x) = x^{s-1}$$

on the condition that

$$\frac{3-s-s^2}{s(1-s)} - \pi \cot \pi s = \frac{8\pi}{3\alpha}.$$

There are two roots in $[0, 1]$ for s , and the condition for a bifurcation is their equality, which occurs when $\alpha = \alpha_c \equiv 0.933\,667$.⁴

The above analysis is adequate precisely at the critical coupling, i.e., at the bifurcation point of the original nonlinear Dyson–Schwinger equation. More generally, in order to avoid infrared divergence difficulties, it is customary to employ the so-called quasilinear equation, which has the form

$$\mathcal{M}(x) = \lambda \int_0^\infty dy \frac{y}{y+1} \left\{ \frac{\theta(x-y)}{x} \left(\mathcal{M}(y) - \frac{y}{2} \frac{\mathcal{M}(y) - \mathcal{M}(x)}{y-x} \right) + \frac{\theta(y-x)}{y} \left(\mathcal{M}(y) - \frac{x}{2} \frac{\mathcal{M}(y) - \mathcal{M}(x)}{y-x} \right) \right\}. \quad (4)$$

This equation is obtained by replacing $\mathcal{M}^2(y)$ in Eq. (1) by $\mathcal{M}^2(0)$, and subsequently scaling this constant to unity. We have set $\lambda = 3\alpha/4\pi$, and the ultraviolet cutoff has been removed ($\Lambda \rightarrow \infty$); we are interested primarily in this article in the mathematical properties of the integral equation (4), rather than in the phenomenon of supercritical chiral symmetry breaking. In the absence of an ultraviolet cutoff, the integral equation has a solution for any value of λ .

The expected power behavior of $\mathcal{M}(x)$ at large x strongly suggests that this equation can be studied by using a Mellin transformation. The Mellin transformation and its inverse are defined by the equations

$$g(s) = \int_0^\infty dx \, x^{s-1} \mathcal{M}(x), \quad (5)$$

$$\mathcal{M}(x) = \frac{1}{2\pi i} \int_L ds \, g(s) x^{-s}, \quad (6)$$

where the contour L goes from $-i\infty$ to $i\infty$. This contour must be chosen appropriately in order to obtain a solution. As we see from Eq. (6), the asymptotic behavior of $\mathcal{M}(x)$ as $x \rightarrow \infty$ is determined by the singularity of $g(s)$ with the *smallest* $\text{Re } s$ to the *right* of L , while the behavior as $x \rightarrow 0$ is determined by the singularity with the *largest* $\text{Re } s$ to the *left* of L . From Eqs. (4)–(6) one finds the transformed integral equation

$$g(s) = \frac{1}{2\pi i} \int_L ds' \, K(s, s') g(s'), \quad (7)$$

where the kernel $K(s, s')$ is given by

$$K(s, s') = \lambda \int_0^\infty dx \int_0^\infty dy \frac{x^{s-1}y}{y+1} \left\{ \frac{\theta(x-y)}{x} \left(y^{-s'} - \frac{y}{2} \frac{y^{-s'} - x^{-s'}}{y-x} \right) + \frac{\theta(y-x)}{y} \left(y^{-s'} - \frac{x}{2} \frac{y^{-s'} - x^{-s'}}{y-x} \right) \right\}. \quad (8)$$

The conditions for the existence of $K(s, s')$ must now be considered. For convergence of the integrals over x and y at both limits, we require

$$\text{as } x \rightarrow 0, \quad \operatorname{Re} s > 0, \quad \text{and as } x \rightarrow \infty, \quad \operatorname{Re} s < 1,$$

$$\text{as } y \rightarrow 0, \quad \operatorname{Re} s' < 2, \quad \text{and as } y \rightarrow \infty, \quad \operatorname{Re} s' > 0,$$

i.e., $0 < \operatorname{Re} s < 1$, $0 < \operatorname{Re} s' < 2$. Changing the integration variable $x = yu$, we obtain

$$K(s, s') = \lambda \int_0^\infty dy \frac{y^{s-s'}}{y+1} \int_0^\infty du u^{s-1} \left\{ \frac{\theta(u-1)}{u} \left(1 - \frac{1}{2} \frac{1-u^{-s'}}{1-u} \right) + \theta(1-u) \left(1 - \frac{u}{2} \frac{1-u^{-s'}}{1-u} \right) \right\} = \frac{\lambda \pi \varphi(s, s')}{\sin \pi(s' - s)} \quad (9)$$

on condition that

$$\operatorname{Re} s < \operatorname{Re} s' < \operatorname{Re}(s+1), \quad (10)$$

where

$$\varphi(s, s') = \frac{1}{s(1-s)} + \frac{1}{2} [\psi(1+s) - \psi(2-s) + \psi(2+s'-s) - \psi(1+s-s')], \quad (11)$$

with

$$\psi(s) = \frac{d \ln \Gamma(s)}{ds}.$$

Note that the integral equation (7) for $g(s)$ is not of the usual type, since it relates $g(s)$ to $g(s')$, where the set of s values is different from the set of s' values, because of the requirement (10). Let us now deform the s' contour so that it half encircles the pole in $K(s, s')$ at $s = s'$. In the neighborhood of this pole

$$K(s, s') \approx \frac{\lambda \varphi(s)}{s' - s},$$

with

$$\varphi(s) \equiv \varphi(s, s) = \frac{3-s-s^2}{2s(1-s)} - \frac{\pi}{2} \cot \pi s \quad (12)$$

we can write

$$K(s, s') = \frac{\lambda \varphi(s)}{s' - s} + \lambda \left[\frac{\pi \varphi(s, s')}{\sin \pi(s' - s)} - \frac{\varphi(s)}{s' - s} \right] \equiv \frac{\lambda \varphi(s)}{s' - s} + k(s, s'), \quad (13)$$

where the kernel $k(s, s')$ is not singular at $s = s'$. Thus we arrive at the following singular integral equation:

$$a(s)g(s) + \frac{b(s)}{\pi i} P \int_L \frac{g(s')}{s' - s} ds' = \frac{1}{2\pi i} \int_L k(s, s')g(s')ds' \equiv g_0(s), \quad (14)$$

where the symbol P denotes a principal value integral, and

$$a(s) = 1 - \frac{\lambda}{2} \varphi(s), \quad b(s) = -\frac{\lambda}{2} \varphi(s).$$

A method for solving singular integral equations of this kind has been extensively discussed in the books by Muskhelishvili and Gakhov.⁷

We shall sketch the method, as it applies to Eq. (14). We first introduce the function

$$\Phi(s) = \frac{1}{2\pi i} \int_L \frac{g(s')ds'}{s' - s}, \quad (15)$$

where the contour L goes from $-i\infty$ to $i\infty$. Clearly $\Phi(s)$ is a single-valued function in the s plane, cut along L . If we denote the region to the left of the contour L by S^+ and that to the right by S^- , then we obtain two functions Φ^+ , Φ^- , analytic in S^+ , S^- , respectively, according to whether s lies in S^+ or S^- . They can be analytically continued beyond L , deforming the contour so long as no singularity of the integrand is encountered, i.e., unless $g(s')$ is singular, or L passes through s . When L does pass through s , the following Sokhotsky–Plemelj formulas are applicable:

$$[\Phi(s - \epsilon) - \Phi(s + \epsilon)]_{\epsilon \rightarrow 0} \equiv \Phi^+(s) - \Phi^-(s) = g(s), \quad (16)$$

$$[\Phi(s - \epsilon) + \Phi(s + \epsilon)]_{\epsilon \rightarrow 0} \equiv \Phi^+(s) + \Phi^-(s) = \frac{P}{\pi i} \int_L \frac{g(s')ds'}{s' - s}. \quad (17)$$

These relations permit us to reduce our integral equation (14) to an algebraic one

$$a(s)[\Phi^+(s) - \Phi^-(s)] + b(s)[\Phi^+(s) + \Phi^-(s)] = g_0(s) \quad (18)$$

or

$$\Phi^+(s) = G(s)\Phi^-(s) + h(s), \quad (19)$$

where

$$G(s) = \frac{a(s) - b(s)}{a(s) + b(s)} = \frac{1}{1 - \lambda \varphi(s)} \quad (20)$$

and

$$h(s) = \frac{g_0(s)}{a(s) + b(s)} = \frac{g_0(s)}{1 - \lambda \varphi(s)}. \quad (21)$$

Finding the analytic function $\Phi(s)$ that satisfies the relation (19) on the contour L is a so-called Hilbert–Riemann problem. Having found the solution, we obtain $g(s)$ through Eq. (16).

According to the general method,^{7,8} we have to consider first the homogeneous version of Eq. (19), namely,

$$\Phi_0^+(s) = G(s)\Phi_0^-(s). \quad (22)$$

Taking the logarithm of both sides, one finds

$$\ln \Phi_0^+(s) - \ln \Phi_0^-(s) = \ln G(s). \quad (23)$$

If one now introduces

$$\ln \Phi_0(s) = \frac{1}{2\pi i} \int_L \frac{\ln G(s') ds'}{s' - s} \quad (24)$$

this would effect a solution of the discontinuity equation (23), if the function $G(s)$ were single-valued on the contour $L[G(\sigma + i\infty) = G(\sigma - i\infty), \sigma = \operatorname{Re} s]$. However, in our case it is easy to calculate that

$$\begin{aligned} G(\sigma + i\infty) &= \frac{2}{\sqrt{(2-\lambda)^2 + \pi^2 \lambda^2}} \exp \left[i \arctan \frac{\lambda \pi}{2-\lambda} \right] \neq \frac{2}{\sqrt{(2-\lambda)^2 + \pi^2 \lambda^2}} \exp \left[-i \arctan \frac{\lambda \pi}{2-\lambda} \right] \\ &= G(\sigma - i\infty). \end{aligned} \quad (25)$$

Accordingly we define the quantity

$$\gamma = \frac{1}{2\pi i} \ln \frac{G(\sigma + i\infty)}{G(\sigma - i\infty)}, \quad (26)$$

where we understand that $G(\sigma + i\infty)$ is obtained from $G(\sigma - i\infty)$ by changing $G(s)$ continuously along the contour L . We pick up the branch of the logarithm in Eq. (26) that satisfies $0 \leq \operatorname{Re} \gamma < 1$. If we put

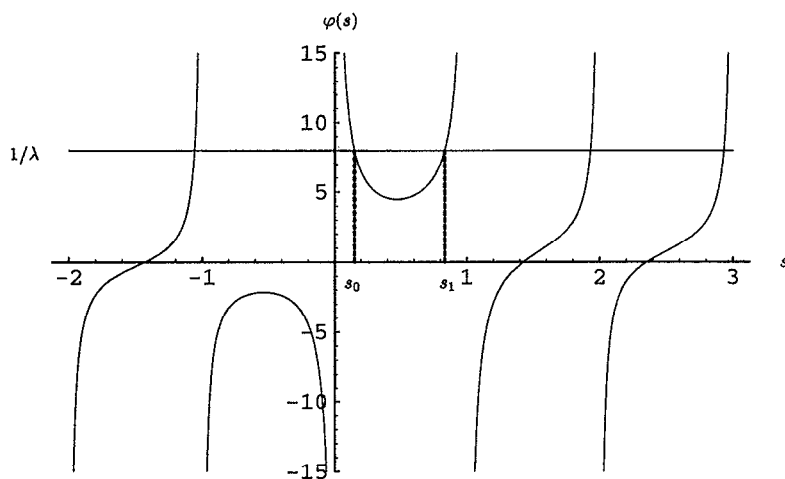


FIG. 1. Graph of $\varphi(x)$.

TABLE I. Poles of $G(s)$ for $0 < s < 1$. Subcritical case.

λ	s_0	s_1
0.05	0.0530	0.9444
0.10	0.1143	0.8749
0.15	0.1914	0.7835
0.20	0.3094	0.6444
0.2229	0.4710	0.4710

$$\frac{G(\sigma + i\infty)}{G(\sigma - i\infty)} = e^{i\theta} \quad (27)$$

then

$$\gamma = \frac{\theta}{2\pi} - \kappa. \quad (28)$$

To satisfy the condition $0 \leq \operatorname{Re} \gamma < 1$ we have to choose the integer κ to be

$$\kappa = \left[\frac{\theta}{2\pi} \right],$$

where $[x]$ denotes the integer part of x . For the case under consideration, $\gamma = (1/\pi) \arctan[\lambda\pi/(2 - \lambda)]$, and so θ and κ depend on the choice of the contour L .

The function $G(s)$ has zeros at $s = n$ (integers) and poles at $s = s_n$ where s_n are roots of the transcendental equation

$$\varphi(s) = 1/\lambda. \quad (29)$$

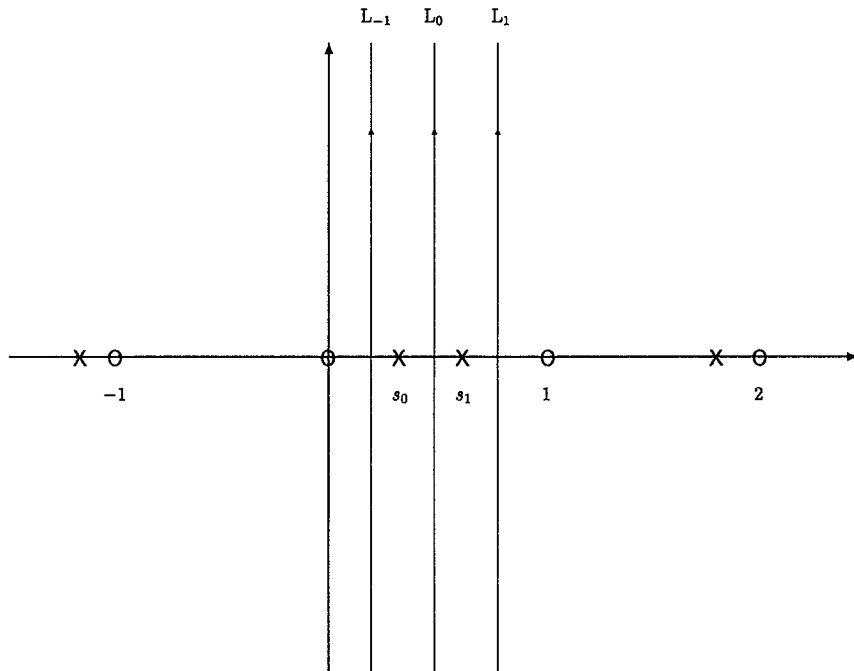
The graph of φ is depicted in Fig. 1: we are of course only interested in the strip $0 < s < 1$. In Table I we give the positions of the two poles of $G(s)$ that lie in the interval $(0, 1)$. We first consider the subcritical domain, $\lambda < \lambda_c$. The last entry in the table, the critical value $\lambda = \lambda_c \equiv 0.222\,896$, corresponds to the minimum of the function $\varphi(s)$.

For larger values of λ , the pole positions are complex conjugates of one another. In Table II we list some results for supercritical λ .

According to the generalized Hadamard theorem, the meromorphic function $G(s)$ can be expressed as an infinite product representation of the form

TABLE II. Poles of $G(s)$ for $0 < s < 1$. Supercritical case.

λ	$\operatorname{Re} s_0$	$\operatorname{Im} s_0$
0.30	0.4454	0.3110
0.40	0.4009	0.4711
0.50	0.3470	0.5807
0.75	0.2040	0.7361
1.00	0.0858	0.7973
2.00	-0.1387	0.8140
10.0	-0.3084	0.7419

FIG. 2. Contours in the complex s plane.

$$G(s) = -\frac{s}{\lambda} \frac{\prod_{n=-\infty}^{\infty} (1 - (s/n)) e^{s/n}}{\prod_{n=-\infty}^{\infty} (1 - (s/s_n)) e^{s/s_n}}, \quad (30)$$

where the prime means omitting the term with $n=0$. For subcritical λ , we have three possible choices for the contour L , which we call L_{-1} , L_0 , and L_1 , as shown in Fig. 2 (poles are represented by X, zeros by O). For larger values, $\lambda > \lambda_c$, the two poles are complex, and in this case we only have two interesting contours, namely, L_{-1} and L_1 .

Let us consider the function $\Omega(s) = ((s-a)/(s-b))^{-\gamma}$, where we take the points a, b to lie on the real axis and to satisfy $a < \operatorname{Re} s = \sigma < b$ (the concrete choice of a, b does not matter, since nothing depends on it). Introducing the function $G_1(s) = \Omega(s)G(s)$ we can show that

$$\frac{G_1(\sigma + i\infty)}{G_1(\sigma - i\infty)} = \frac{\Omega(\sigma + i\infty)G(\sigma + i\infty)}{\Omega(\sigma - i\infty)G(\sigma - i\infty)} = e^{-2\pi i \gamma} \frac{G(\sigma + i\infty)}{G(\sigma - i\infty)} = 1. \quad (31)$$

In terms of the new functions $\Phi_0^+(s) = \Phi_1^+(s)(s-b)^{-\gamma}$, $\Phi_0^-(s) = \Phi_1^-(s)(s-a)^{-\gamma}$, Eq. (22) is reduced to the homogeneous Riemann problem

$$\Phi_1^+(s) = G_1(s)\Phi_1^-(s), \quad s \in L, \quad (32)$$

where $G_1(s)$ is a continuous function on the contour L , including the point at infinity [cf. condition (31)]. The functions $\omega^+(s) = (s-b)^{\gamma}$, $\omega^-(s) = (s-a)^{\gamma}$, have branch points at (b, ∞) and (a, ∞) , respectively. If we make a cut in the s plane going from the point $s=a$ up to the point $s=b$, passing through the point at infinity on L , then the functions $\omega^+(s)$ and $\omega^-(s)$ are single valued in such a plane. Under these conditions, $\omega^+(s)$ will be analytic in S^+ , and $\omega^-(s)$ analytic in S^- . Thus we have to find the analytic function $\Phi_1(s)$ that satisfies the condition (32) on L .

We shall compute the index of the problem (32)

$$\text{Ind } G_1(s) \equiv \frac{1}{2\pi i} \Delta_L \ln G_1(s) = \frac{1}{2\pi i} \ln \left(\frac{G(\sigma + i\infty)}{G(\sigma - i\infty)} e^{-2\pi i \gamma} \right) = \frac{1}{2\pi i} \ln e^{2\pi i \kappa} = \kappa, \quad (33)$$

where the notation Δ_L is meant to include the total change of the function as we traverse the entire contour L . The index κ determines the branches of $\ln G_1(s)$ and $\ln G(s)$. If $\kappa \neq 0$ we must consider the function $((s-a)/(s-b))^{-\kappa} G_1(s)$. We define the analytic function

$$\Gamma(s) = \frac{1}{2\pi i} \int_L \frac{\ln[(s'-a)/(s'-b)]^{-\kappa} G_1(s') ds'}{s'-s} \quad (34)$$

so that

$$X(s) = X^+(s) = \exp \Gamma^+(s), \quad s \in S^+,$$

$$X(s) = X^-(s) = \left(\frac{s-a}{s-b} \right)^{-\kappa} \exp \Gamma^-(s), \quad s \in S^-$$

yields the solution of Eq. (32). In fact

$$\frac{X^+(s)}{X^-(s)} = \left(\frac{s-a}{s-b} \right)^{\kappa} \exp[\Gamma^+(s) - \Gamma^-(s)] = G_1(s), \quad s \in L. \quad (35)$$

Next we consider the inhomogeneous Hilbert–Riemann problem (19), which can be written in terms of the functions Φ_1^+, Φ_1^- as

$$\Phi_1^+(s) = G_1(s) \Phi_1^-(s) + (s-b)^{\gamma} h(s). \quad (36)$$

Using the factorization $G_1(s) = X^+(s)/X^-(s)$, we obtain from Eq. (36)

$$\frac{\Phi_1^+(s)}{X^+(s)} - \frac{\Phi_1^-(s)}{X^-(s)} = \frac{(s-b)^{\gamma} h(s)}{X^+(s)}. \quad (37)$$

The final problem consists in finding a function with a discontinuity given on the contour L . We introduce the analytic function

$$\Psi(s) = \frac{1}{2\pi i} \int_L \frac{(s'-b)^{\gamma} h(s') ds'}{X^+(s')(s'-s)}. \quad (38)$$

From Eq. (38) it is clear why we have to have the constraint $\text{Re } \gamma < 1$. With the help of $\Psi(s)$ we can rewrite Eq. (37) in the form

$$\frac{\Phi_1^+(s)}{X^+(s)} - \Psi^+(s) = \frac{\Phi_1^-(s)}{X^-(s)} - \Psi^-(s), \quad s \in L, \quad (39)$$

which defines a function analytic at all points except possibly the pole at $s=b$ (in the case $\kappa > 0$). Thus the solution of Eq. (39) that vanishes at infinity is given by

$$\Phi_1(s) = X(s) \left[\Psi(s) - \frac{1}{2} \frac{P_{\kappa-1}(s)}{(s-b)^{\kappa}} \right] = \Phi_1^+(s), \quad \text{for } \text{Re } s < \sigma \quad (40)$$

and

$$\Phi_1(s) = \Phi_1^-(s), \quad \text{for } \operatorname{Re} s > \sigma.$$

Here

$$X(s) = X^+(s), \quad \text{for } \operatorname{Re} s < \sigma \quad \text{and} \quad X(s) = X^-(s), \quad \text{for } \operatorname{Re} s > \sigma, \quad (41)$$

while $P_{\kappa-1}(s)$ is a polynomial of degree $(\kappa-1)$ [for $\kappa \leq 0$ we set $P_{\kappa-1}(s) \equiv 0$, and the coefficient $-\frac{1}{2}$ is introduced for later convenience]. For the function $g(s) = \Phi^+(s) - \Phi^-(s)$ we obtain, after some algebraic manipulations

$$g(s) = \frac{1}{a(s)+b(s)} \left[a(s)g_0(s) - \frac{b(s)Z(s)}{\pi i} \int_L \frac{g_0(s')ds'}{Z(s')(s'-s)} + \frac{b(s)Z(s)P_{\kappa-1}(s)}{(s-b)^\kappa} \right], \quad (42)$$

where the function $Z(s)$ is given by

$$Z(s) = [a(s)+b(s)]X^+(s)(s-b)^{-\gamma} = (s-a)^{-\gamma}X^-(s). \quad (43)$$

Taking into account the notation (14) for $g_0(s)$, we arrive at the following integral equation:

$$g(s) + \int_L \mathcal{K}(s,s')g(s')ds' = \frac{b(s)Z(s)}{a(s)+b(s)} \frac{P_{\kappa-1}(s)}{(s-b)^\kappa} \equiv \Psi_0(s), \quad (44)$$

with the kernel

$$\mathcal{K}(s,s') = -\frac{a(s)k(s,s')}{a(s)+b(s)} + \frac{b(s)Z(s)}{a(s)+b(s)} \frac{1}{\pi i} \int_L \frac{k(s'',s')ds''}{Z(s'')(s''-s)}. \quad (45)$$

It is not difficult to verify that the kernel $\mathcal{K}(s,s')$ is not singular at $s=s'$, so Eq. (44) is Fredholm. For the values λ not coinciding with the eigenvalues of Eq. (44), there exists a unique solution of the form

$$g(s) = \Psi_0(s) - \int_L R(s,s';\lambda)\Psi_0(s')ds', \quad (46)$$

where $R(s,s';\lambda)$ is the resolvent of Eq. (44), which can be expressed in terms of the kernel $\mathcal{K}(s,s';\lambda)$, according to the general theory of Fredholm equations.

From the solution of the Riemann problem, we saw that in the case that $\kappa \leq 0$, we have to put $P_{\kappa-1} \equiv 0$, and we obtain only the trivial solution, $g(s) = 0$. Thus in order to obtain a nontrivial solution, we must choose our contour L in such a way that $\kappa > 0$. Among the three contours L_{-1}, L_0, L_1 shown in Fig. 2, only L_1 satisfies this condition, as we will show at the end of this article.

The analytic structure in the complex s plane of the solution $g(s)$, as given in Eq. (46), is determined entirely by that of the function $\Psi_0(s)$,

$$\Psi_0(s) = C(s-b)^{-\gamma-1}b(s)X^+(s) = \frac{C(s-a)^{-\gamma}}{s-b} \frac{b(s)}{a(s)+b(s)} X^-(s) \quad (47)$$

or

$$\Psi_0(s) = -\frac{\lambda}{2} C\varphi(s)(s-b)^{-\gamma-1}e^{\Gamma^+(s)}$$

$$= -\frac{\lambda}{2} \frac{C\varphi(s)}{1-\lambda\varphi(s)} (s-a)^{-\gamma-1} e^{\Gamma^-(s)}. \quad (48)$$

It is convenient to use two forms for writing $\Psi_0(s)$ in the regions S^+ and S^- . Since $\Psi_0(s)$ depends on one arbitrary constant C , the solution $g(s)$ depends on one arbitrary constant too [that is clear, since the original equation (4) is homogeneous].

To the right of the contour L , the function $\Psi_0(s)$ has poles at solutions of the equation $\varphi(s)=1/\lambda$, i.e., at $s=s_n$ [the poles of the function $\varphi(s)$ at $s=n$ are canceled, see Eq. (48)]. So the leading asymptotic behavior of $f(x)$ is given by

$$f(x) \sim x^{-s_0}, \quad x \rightarrow \infty. \quad (49)$$

To the left of the contour L , when $x \rightarrow 0$, the asymptotic behavior of $f(x)$ is determined by the singularity of $\varphi(s)$ at $s=0$. Thus we have

$$f(x) \sim \text{const}, \quad x \rightarrow 0. \quad (50)$$

There is in fact another way to reveal the analytic structure of $g(s)$. If one knows that the solution of the Riemann problem (19)–(21) indeed exists, then the function $g(s)$ can be written in terms of $\Phi^+(s)$, $\Phi^-(s)$, which are analytic in the regions S^+ , S^- , respectively. For example, for $s \in S^-$ we write

$$g(s) = \Phi^+(s) - \Phi^-(s) = G(s)[\Phi^-(s) + g_0(s)] - \Phi^-(s). \quad (51)$$

Thus $g(s)$ will have poles at the poles of function $G(s)$, i.e., at the points $s=s_n$. By construction we know that $g_0(s)$ has poles at $s=n$, too, but here $G(s)$ has zeros that cancel these singularities. The asymptotic behavior of $\mathcal{M}(x)$ will be dominated by the smallest s_n^{\min} in the S^- region, i.e.,

$$\mathcal{M}(x) \sim x^{-s_n^{\min}}, \quad x \rightarrow \infty. \quad (52)$$

For $s \in S^+$ we can write

$$g(s) = \Phi^+(s) - \frac{\Phi^+(s)}{G(s)} + g_0(s). \quad (53)$$

The singularities of $g(s)$ in S^+ are situated either at zeros of $G(s)$ or at poles of $g_0(s)$, or at both. Thus there may be poles of $g(s)$ at all integers to the left of L . If the contour L_1 is chosen, there will be a pole in $g(s)$ at $s=0$ and $f(x)$ will have the behavior

$$f(x) \sim \text{const}, \quad x \rightarrow 0. \quad (54)$$

It is clear that the transition from nonoscillatory to oscillatory behavior of $f(x)$ at large x occurs when two real roots, s_0 and s_1 , of $\varphi(s)=1/\lambda$, fuse together, and then become complex conjugate. This determines the critical coupling constant, according to

$$\frac{1}{\lambda_c} = \min \varphi(s), \quad 0 < s < 1 \quad (55)$$

and in our case this has the form

$$\frac{1}{\lambda_c} = \min \left[\frac{3-s-s^2}{2s(1-s)} - \frac{\pi}{2} \cot \pi s \right], \quad 0 < s < 1. \quad (56)$$

It is evident from Eq. (56) that the minimum of $\varphi(s)$ is reached at a point $s \neq 1/2$. In fact $s_{\min} < 1/2$, which means that the anomalous dimension $\gamma_m > 1$ ($\gamma_m = 2 - 2s$), contradicts Holdom's claims that in quenched quantum electrodynamics in four dimensions (QED₄) we always have $\gamma_m = 1$.⁹ The reason for this disagreement can perhaps be sought in the fact that Holdom's arguments are based on the finite order skeleton approximation in quenched QED, whereas the Curtis–Pennington and the Bashir–Pennington *Ansätze* may more adequately represent the infinite sum of such terms. This question is worthy of further investigation, for example, studying the vertex equation itself by a higher-order Dyson–Schwinger equation in order to investigate its solution in Curtis–Pennington form.

In conclusion we shall discuss the computation of the index κ of the Riemann problem (19). Let us write the function $\varphi(s = \sigma + it)$ in the form $\varphi(s) = u(\sigma, t) + iv(\sigma, t)$, where the real and imaginary parts are given by

$$u(\sigma, t) = \frac{1}{2} \frac{(3 - \sigma - \sigma^2)\sigma(1 - \sigma) + 2t^2(1 + \sigma^2) + t^4}{[\sigma(1 - \sigma) + t^2]^2 + t^2(1 - 2\sigma)^2} - \frac{\pi}{2} \frac{\sin 2\pi\sigma}{\cosh 2\pi t - \cos 2\pi\sigma}, \quad (57)$$

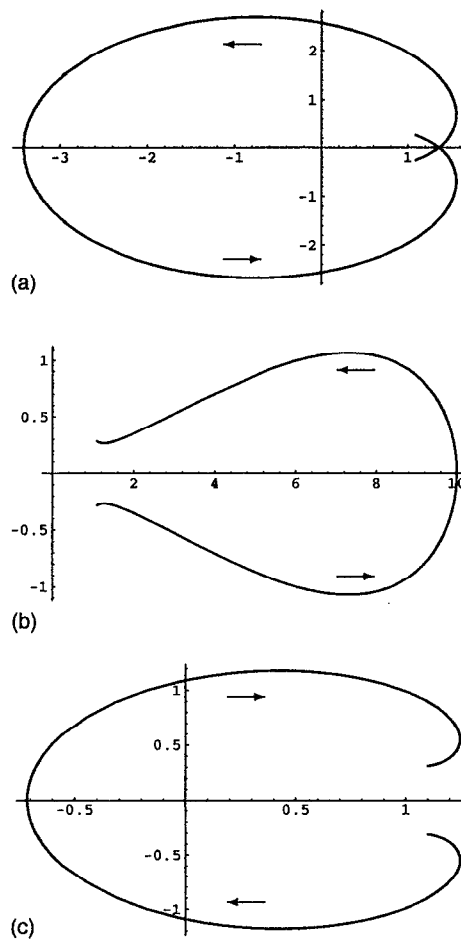


FIG. 3. Plot in the complex plane of $G(s)$, $s = \sigma + it$, $\{t, -\infty, \infty\}$. (a) $G(s)$, $s \in L_1$. (b) $G(s)$, $s \in L_0$. (c) $G(s)$, $s \in L_{-1}$.

$$v(\sigma, t) = -\frac{t}{2} \frac{2\sigma^2 - 6\sigma + 3 + 2t^2}{[\sigma(1-\sigma) + t^2]^2 + t^2(1-2\sigma)^2} + \frac{\pi}{2} \frac{\sinh 2\pi t}{\cosh 2\pi t - \cos 2\pi\sigma}.$$

Writing

$$G(s) = \frac{1}{1 - \lambda u - i\lambda v} = \frac{1 - \lambda u + i\lambda v}{(1 - \lambda u)^2 + \lambda^2 v^2} \equiv \rho(\sigma, t) e^{i\theta(\sigma, t)},$$

we obtain

$$\rho(\sigma, t) = |G(s)| = \frac{1}{\sqrt{(1 - \lambda u)^2 + \lambda^2 v^2}} \quad (58)$$

and

$$\theta(\sigma, t) = \arctan \frac{\lambda v}{1 - \lambda u}. \quad (59)$$

Now $\rho(\sigma, t = \infty) = \rho(\sigma, t = -\infty)$, and we have from Eq. (27)

$$\theta = \theta(\sigma, t = +\infty) - \theta(\sigma, t = -\infty). \quad (60)$$

If we define the argument of $G(s)$ at $s = \sigma - i\infty$ to be $\theta(\sigma, t = -\infty) = -\arctan[\lambda\pi/(2-\lambda)]$ [from Eq. (57) $u(\sigma, t = -\infty) = 1/2$, $v(\sigma, t = -\infty) = -\pi/2$], then we have to follow the change of $\theta(\sigma, t)$ along the contour L . In fact, we know the value $\theta(\sigma, t = +\infty) = \arctan[\lambda\pi/(2-\lambda)]$, apart from the part connected with a possible winding around the point $G=0$ in the complex G plane, which is equal to $2\pi n$ (n being an integer). The integer n (or, connected with it, κ) can be calculated numerically by simply plotting the graph of G (see Fig. 3). Having the plot of G , the winding number κ can be determined visually for different contours L .

According to general theory, the index is defined by the formula $\kappa = [\theta/2\pi]$, where θ is determined from Eq. (27). In Fig. 3(a), corresponding to contour L_1 of Fig. 2, one sees that, since the curve is traversed in a counterclockwise—i.e., a positive—sense, as t goes from $-\infty$ to ∞ , one has $2\pi < \theta < 4\pi$, which implies $\kappa = 1$. Hence in this case we have a solution of the Riemann–Hilbert problem, as we claimed above. Figure 3(b) corresponds to the contour L_0 , which again gives a positive θ , but one less than 2π , so here $\kappa = 0$. Finally, in Fig. 3(c) we show the situation with the contour L_{-1} , where the curve is described in the negative sense, but does not make a complete rotation around the origin. In this case, $-2\pi < \theta < 0$, so $\kappa = -1$. In the last two cases one has in general *not* generated a solution of the problem.

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